Global Existence of Strong Solutions to Incompressible MHD *

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Abstract

We establish the global existence and uniqueness of strong solutions to the initial boundary value problem for incompressible MHD equations in a bounded smooth domain of three spatial dimensions with initial density being allowed to have vacuum, in particular, the initial density can vanish in a set of positive Lebessgue measure. More precisely, under the assumption that the production of the quantities $\|\sqrt{\rho}_0 u_0\|_{L^2(\Omega)}^2+\|H_0\|_{L^2(\Omega)}^2$ and $\|\nabla u_0\|_{L^2(\Omega)}^2+\|\nabla H_0\|_{L^2(\Omega)}^2$ is suitably small, with the smallness depending only on the bound of the initial density and the domain, we prove that there is a unique strong solution to the Dirichlet problem of the incompressible MHD system.

Keywords: Incompressible MHD; global existence and uniqueness; strong solutions.

1 Introduction

Magnetohydrodynamics (MHD for short) is the study of the interaction between magnetic fields and moving conducting fluids, which can be described by the following system

$$\rho_t+\operatorname{div}(\rho u)=0, \tag{1.1}$$

$$\rho(u_t+(u\cdot\nabla)u)-\mu\Delta u+\nabla p=(\nabla\times H)\times H, \tag{1.2}$$

$$H_t-\lambda\Delta H=\nabla\times(u\times H), \tag{1.3}$$

$$\operatorname{div}u=\operatorname{div}H=0, \tag{1.4}$$

where $\rho=\rho(x,t)\in\mathbb{R}^+$ denotes the density, $u=u(x,t)\in\mathbb{R}^3$ the fluid velocity, $H=H(x,t)\in\mathbb{R}^3$ the magnetic field, $p=p(x,t)\in\mathbb{R}$ the pressure, positive constant μ is called the kinematic viscosity and positive constant λ is called the magnetic diffusivity. Usually,

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we refer to equation (1.1) as the continuity equation, which represents the conservation lass of the mass, to (1.2) as the momentum conservation equation. Equation (1.3) is a considerably simplified version of the well known Maxiwell's equations (see [5, 1, 2]) by dropping Gauss's law and ignoring the displacement currents, and it's sometimes called the induction equation. As for the constraint $\mathrm{div} H = 0$, it can be seen just as a restriction on the initial value H_0 of H since $\mathrm{div} H_t = 0$. Note that, using the condition (1.4), equations (1.2) and (1.3) can be rewritten as

$$\rho(u_t + (u \cdot \nabla)u) - \mu\Delta u + \nabla p = (H \cdot \nabla)H,$$
$$H_t + (u \cdot \nabla)H - \lambda\Delta H = (H \cdot \nabla)u.$$

There are a lot of literatures on the study of MHD. The global existence of weak solutions to the homogeneous incompressible MHD was proven in [3] and [4] long time ago, the density dependent case was later proven in [5] by using Lions's method [6]. The corresponding results on the global existence of weak solutions to the compressible MHD are proven in [8, 9, 13, 10] by using the method exploited in [7] (see also [11, 12]), where in the first paper the isentropic case is considered, while in the other three ones concern the non-isothermal model. Local existence and uniqueness of strong solutions can be found in [14] and [15] for incompressible model, in [17] and [18] for the compressible model, where in the last three papers the non-isothermal model are considered. Global existence and uniqueness of strong solutions to the incompressible MHD with vacuum in two space dimensions is proved in [16], recently, and this work is new even for the inhomogeneous incompressible Navier-Stokes equations. While the three dimensional case, with small initial data and away from vacuum, is proven in [19].

One of the most important questions for system (1.1)–(1.4) is to prove the global existence and uniqueness of solutions (ρ, u, p) satisfying the initial condition

$$(\rho, u, H)|_{t=0} = (\rho_0, u_0, H_0) \quad \text{in } \Omega, \tag{1.5}$$

and boundary condition

$$u|_{\partial\Omega} = 0, \quad H \cdot n|_{\partial\Omega} = 0, \quad (\nabla \times H) \times n|_{\partial\Omega} = 0, \tag{1.6}$$

where n is the unit outward norm vector on $\partial\Omega$.

The aim of this paper is to prove the global existence and uniqueness of strong solutions to system (1.1)–(1.6) with the initial data being allowed to have vacuum. For $1 \le p \le \infty$, we denote by $\|u\|_p$ the $L^p(\Omega)$ norm of the function u. The definition of strong solution is stated in the following:

Definition 1.1. *We call (ρ, u, H) a strong solution to the system (1.1)–(1.6) on $(0, T)$ if (ρ, u, H) satisfies (1.1)–(1.4) a.e. in $\Omega \times (0, T)$ for some pressure function p, and satisfies the initial condition (1.5) and boundary condition (1.6), with the regularity*

$$\rho \in L^\infty(Q_T) \cap L^\infty(0, T; H^1(\Omega)), \quad \rho_t \in L^\infty(0, T; L^2(\Omega)),$$
$$u \in L^\infty(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \cap L^2(0, T; W^{2,6}(\Omega)),$$
$$H \in L^\infty(0, T; H^2(\Omega)) \cap L^2(0, T; W^{2,6}(\Omega)),$$
$$u_t, H_t \in L^2(0, T; H^1(\Omega)), \quad \sqrt{\rho}u_t, H_t \in L^\infty(0, T; L^2(\Omega)).$$

Our main result is stated bellow:

Theorem 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^3$ and $\bar{\rho}$ a positive number. Assume that the initial data (ρ_0, u_0) satisfies the conditions*

$$0 \le \rho_0 \le \bar{\rho}, \quad \rho_0 \in H^1(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad H_0 \in H^2(\Omega)$$

and the compatibility condition

$$\begin{aligned} &div u_0 = div H_0 = 0, \quad H_0 \cdot n|_{\partial\Omega} = 0, \quad (\nabla \times H_0) \times n|_{\partial\Omega} = 0, \\ &\Delta u_0 + (H_0 \cdot \nabla) H_0 - \nabla p_0 = \sqrt{\rho_0} g_0 \quad in\ \Omega \end{aligned}$$

for some $(p_0, g_0) \in H^1(\Omega) \times L^2(\Omega)$. Then there exists a positive constant ε_0 depending only on $\bar{\rho}$ and Ω, such that, if

$$(\|\sqrt{\rho_0} u_0\|_2^2 + \|H_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\nabla H_0\|_2^2) \le \varepsilon_0, \tag{1.7}$$

then the system (1.1)–(1.6) has a unique global strong solution.

Remark 1.1. *The quantity $(\|\sqrt{\rho} u\|_2^2 + \|H\|_2^2)(\|\nabla u\|_2^2 + \|\nabla H\|_2^2)$ is scaling invariable under the scaling transform*

$$\begin{aligned} &\rho_\lambda(x,t) = \rho(\lambda x, \lambda^2 t), \quad p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t), \\ &u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t), \quad H_\lambda(x,t) = \lambda H(\lambda x, \lambda^2 t), \end{aligned}$$

and thus Theorem 1.1 can be viewed as a result on the global existence of strong solutions with vacuum in critical space.

2 Proof of Theorem 1.1

Throughout this section, we denote

$$\|\sqrt{\rho_0} u_0\|_2^2 + \|H_0\|_2^2 = C_0^2.$$

Definition 2.1. *A finite time T_* is called the finite blow-up time if*

$$\Phi(T) < \infty \quad for\ all \quad 0 \le T < T_* \quad and \quad \lim_{T \to T_*} \Phi(T) = \infty,$$

where the function $\Phi(T)$ is given by

$$\begin{aligned} \Phi(T) = &\sup_{0 \le t \le T} (\|(\nabla\rho, \rho_t)\|_2 + \|(u, H)\|_{H^2} + \|H_t\|_2 + \|\sqrt{\rho} u_t\|_2) \\ &+ \int_0^T (\|(u, H)\|_{W^{2,6}}^2 + \|(u_t, H_t)\|_{H^1}^2) dt. \end{aligned}$$

We will use the following lemma, which states the local existence and blow up criterion of local strong solutions.

Lemma 2.1. *(See [14]) Under the conditions of Theorem 1.1, there hold*

(i) (Local existence) there exists a small time T_* and a unique strong solution on $(0,T_*)$,

(ii) (Blow-up criterion) T_* is the finite blow-up time of (ρ,u,H) if and only if

$$\int_0^T \|(\nabla u,\nabla H)\|_2^8 dt<\infty \quad \text{for all} \quad 0<T<T_*, \quad \text{and} \quad \int_0^{T_*} \|(\nabla u,\nabla H)\|_2^8 dt=\infty.$$

To proof the global existence of strong solutions, we need to extend the local strong solution given in the above lemma to be a global one. For this purpose, we need do some a priori estimates on the local strong solutions. The following two lemmas give the energy estimates on the strong solutions, where the first one concerns the basic energy estimates, while the second one concerns the higher order estimates.

Lemma 2.2. *Let (ρ,u,H) be a strong solution to system (1.1)–(1.6) on $(0,T)$. Then, there holds*

$$(\|\sqrt{\rho}u\|_2^2+\|H\|_2^2)(t)+2\int_0^t(\|\nabla u\|_2^2+\|\nabla H\|_2^2)ds=\|\sqrt{\rho_0}u_0\|_2^2+\|H_0\|_2^2$$

for any $0\le t\le T$.

Proof. Multiply (1.2) by u and integrate over Ω, by the aid of (1.1), we obtain after integration by parts that

$$\frac{d}{dt}\int_\Omega \frac{\rho}{2}|u|^2dx+\int_\Omega|\nabla u|^2dx=\int_\Omega(H\cdot\nabla H)\cdot udx=-\int_\Omega(H\cdot\nabla)u\cdot Hdx. \tag{2.1}$$

Multiply (1.3) by H and integrate over Ω, we obtain after integration by parts that

$$\frac{d}{dt}\int_\Omega\frac{|H|^2}{2}dx+\int_\Omega|\nabla H|^2dx=\int_\Omega(H\cdot\nabla)u\cdot Hdx. \tag{2.2}$$

Summing (2.1) with (2.2) up, and integrating the resulting equation with respect to t, we obtain

$$\int_\Omega(\rho|u|^2+|H|^2)dx+2\int_0^t\int_\Omega(|\nabla u|^2+|\nabla H|^2)dxds=\int_\Omega(\rho_0|u_0|^2+|H_0|^2)dx,$$

completing the proof. □

Lemma 2.3. *Let (ρ,u,H) be a strong solution to system (1.1)–(1.6) on $(0,T)$. Then, there holds*

$$\begin{aligned}&\sup_{0\le s\le t}(\|\nabla u\|_2^2+\|\nabla H\|_2^2)+\int_0^t(\|\nabla^2u\|_2^2+\|\nabla^2H\|_2^2)ds\\ \le&2(\|\nabla u_0\|_2^2+\|\nabla H_0\|_2^2)+C\sup_{0\le s\le t}(C_0^2\|\nabla u\|_2^4+C_0\|\nabla H\|_2^3)\\ &+CC_0\sup_{0\le s\le t}(\|\nabla u\|_2+\|\nabla H\|_2)\int_0^t(\|\nabla^2u\|_2^2+\|\nabla^2H\|_2^2)ds\end{aligned}$$

for any $t\in(0,T)$.

Proof. Multiplying (1.1) by u_t and integration by parts yields

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{2}dx + \int_\Omega \rho|u_t|^2dx \\
=&\int_\Omega [(H\cdot\nabla)H\cdot u_t - \rho(u\cdot\nabla)u\cdot u_t]dx = -\int_\Omega [\rho(u\cdot\nabla)u\cdot u_t + (H\cdot\nabla)u_t\cdot H]dx \\
=&-\frac{d}{dt}\int_\Omega (H\cdot\nabla)u\cdot Hdx + \int_\Omega [(H_t\cdot\nabla)u\cdot H + (H\cdot\nabla)u\cdot H_t - \rho(u\cdot\nabla)u\cdot u_t]dx \\
\le&-\frac{d}{dt}\int_\Omega (H\cdot\nabla)u\cdot Hdx + \int_\Omega \left(\frac{1}{2}\rho|u_t|^2 + \frac{1}{4}|H_t|^2\right)dx + 4\int_\Omega (|H|^2|\nabla u|^2 + \rho|u|^2|\nabla u|^2)dx,
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(\frac{|\nabla u|^2}{2} + (H\cdot\nabla)u\cdot H\right)dx + \int_\Omega \rho|u_t|^2dx \\
\le&\int_\Omega \left(\frac{1}{2}\rho|u_t|^2 + \frac{1}{4}|H_t|^2\right)dx + 4\int_\Omega (|H|^2|\nabla u|^2 + \rho|u|^2|\nabla u|^2)dx. \qquad (2.3)
\end{aligned}
$$

Multiplying (1.3) by H_t and integration by parts yields

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{|\nabla u|^2}{2}dx + \int_\Omega |H_t|^2dx = \int_\Omega [(H\cdot\nabla)u - (u\cdot\nabla)H]H_tdx \\
\le&\frac{1}{4}\int_\Omega |H_t|^2dx + \int_\Omega (|H|^2|\nabla u|^2 + |u|^2|\nabla H|^2)dx. \qquad (2.4)
\end{aligned}
$$

Summing (2.3) with (2.4) up, it follows

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (|\nabla u|^2 + |\nabla H|^2 + 2(H\cdot\nabla)u\cdot H)dx + \int_\Omega (\rho|u_t|^2 + |H_t|^2)dx \\
\le&10\int_\Omega (|H|^2|\nabla u|^2 + |u|^2|\nabla H|^2 + \rho|u|^2|\nabla u|^2)dx,
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\sup_{0\le s\le t}(\|\nabla u\|_2^2 + \|\nabla H\|_2^2) + \int_0^t (\|\sqrt{\rho}u_t\|_2^2 + \|H_t\|_2^2)ds \\
\le&(\|\nabla u_0\|_2^2 + \|\nabla H_0\|_2^2) + 4\sup_{0\le s\le t}\int_\Omega |H|^2|\nabla u|dx \\
&+ 10\int_0^t\int_\Omega (|H|^2|\nabla u|^2 + |u|^2|\nabla H|^2 + \rho|u|^2|\nabla u|^2)dxds. \qquad (2.5)
\end{aligned}
$$

Applying H^2 estimates to Stokes equations and elliptic equations, it follows from (1.2) and (1.3) that

$$
\|\nabla^2 u\|_2^2 \le C(\|\rho u_t\|_2^2 + \|\rho(u\cdot\nabla)u\|_2^2 + \|(H\cdot\nabla)H\|_2^2)
$$

$$\leq C(\|\sqrt{\rho}u_t\|_2^2 + \|\sqrt{\rho}(u\cdot\nabla)u\|_2^2 + \|(H\cdot\nabla)H\|_2^2) \tag{2.6}$$

and

$$\|\nabla^2 H\|_2^2 \leq C(\|H_t\|_2^2 + \|(u\cdot\nabla)H\|_2^2 + \|(H\cdot\nabla)u\|_2^2), \tag{2.7}$$

where we have used the fact that $0 \leq \rho \leq \bar{\rho}$. Combining (2.5) with (2.6), together with (2.7), we obtain

$$\begin{aligned}
&\sup_{0\leq s\leq t}(\|\nabla u\|_2^2 + \|\nabla H\|_2^2) + \int_0^t (\|\sqrt{\rho}u_t\|_2^2 + \|\nabla^2 u\|_2^2 + \|H_t\|_2^2 + \|\nabla^2 H\|_2^2)ds \\
\leq&(\|\nabla u_0\|_2^2 + \|\nabla H_0\|_2^2) + 4\sup_{0\leq s\leq t}\int_\Omega |H|^2|\nabla u|dx \\
&+ C\int_0^t\int_\Omega [|H|^2(|\nabla u|^2 + |\nabla H|^2) + |u|^2|\nabla H|^2 + \rho|u|^2|\nabla u|^2]dxds.
\end{aligned} \tag{2.8}$$

It follows from Hölder inequality, Sobolev inequality and Young inequality that

$$\begin{aligned}
\int_\Omega |H|^2|\nabla u|dx \leq&\|H\|_4^2\|\nabla u\|_2 \leq \|H\|_2^{1/2}\|H\|_6^{3/2}\|\nabla u\|_2 \\
\leq&C\|H\|_2^{1/2}\|\nabla H\|_2^{3/2}\|\nabla u\|_2 \leq \varepsilon\|\nabla u\|_2^2 + C\|H\|_2\|\nabla H\|_2^3, \\
\int_\Omega \rho|u|^2|\nabla u|^2dx \leq&C\|\sqrt{\rho}u\|_2\|u\|_6\|\nabla u\|_6^2 \leq C\|\sqrt{\rho}u\|_2\|\nabla u\|_2\|\nabla^2 u\|_2^2, \\
\int_\Omega |u|^2|\nabla H|^2dx \leq&\|u\|_6^2\|\nabla H\|_3^2 \leq C\|\nabla u\|_2^2\|\nabla H\|_2\|\nabla H\|_6 \\
\leq&C\|\nabla u\|_2^2\|\nabla H\|_2\|\nabla^2 H\|_2 \leq \varepsilon\|\nabla^2 H\|_2^2 + C\|\nabla u\|_2^4\|\nabla H\|_2^2, \\
\int_\Omega |H|^2(|\nabla u|^2 + |\nabla H|^2)de \leq&\|H\|_2\|H\|_6(\|\nabla u\|_6^2 + \|\nabla H\|_6^2) \\
\leq&C\|H\|_2\|\nabla H\|_2(\|\nabla^2 u\|_2^2 + \|\nabla^2 H\|_2^2).
\end{aligned}$$

Substituting the above inequalities into (2.8), taking ε small enough, it follows from Lemma 2.2 that

$$\begin{aligned}
&\sup_{0\leq s\leq t}(\|\nabla u\|_2^2 + \|\nabla H\|_2^2) + \int_0^t (\|\sqrt{\rho}u_t\|_2^2 + \|\nabla^2 u\|_2^2 + \|H_t\|_2^2 + \|\nabla^H\|_2^2)ds \\
\leq&2(\|\nabla u_0\|_2^2 + \|\nabla H_0\|_2^2) + C\sup_{0\leq s\leq t}\|H\|_2\|\nabla H\|_2^3 + C\sup_{0\leq s\leq t}(\|\sqrt{\rho}u\|_2\|\nabla u\|_2 + \|H\|_2\|\nabla H\|_2) \\
&\times\int_0^t (\|\nabla^2 u\|_2^2 + \|\nabla^2 H\|_2^2)ds + C\sup_{0\leq s\leq t}\|\nabla u\|_2^4\int_0^t \|\nabla H\|_2^2 ds \\
\leq&2(\|\nabla u_0\|_2^2 + \|\nabla H_0\|_2^2) + C\sup_{0\leq s\leq t}(C_0^2\|\nabla u\|_2^4 + C_0\|\nabla H\|_2^3) \\
&+ CC_0\sup_{0\leq s\leq t}(\|\nabla u\|_2 + \|\nabla H\|_2)\int_0^t (\|\nabla^2 u\|_2^2 + \|\nabla^2 H\|_2^2)ds,
\end{aligned}$$

completing the proof. □

By the aid of the above two lemmas, we can prove the following a priori estimates.

Lemma 2.4. *Let (ρ, u, H) be a strong solution to system (1.1)–(1.6) on $(0,T)$. Then, there exists a positive constant ε_0 depending only on $\bar\rho$ and Ω, such that*

$$\sup_{0\le t\le T}(\|\nabla u\|_2^2+\|\nabla H\|_2^2)+\int_0^T(\|\nabla^2 u\|_2^2+\|\nabla^H\|_2^2)dt\le 4(\|\nabla u_0\|_2^2+\|\nabla H_0\|_2^2),$$

provided

$$C_0(\|\nabla u_0\|_2^2+\|\nabla H_0\|_2^2)\le\varepsilon_0.$$

Proof. Define functions $E(t)$ and $\Phi(t)$ as follows

$$E(t)=\sup_{0\le s\le t}(\|\nabla u\|_2^2+\|\nabla H\|_2^2)+\int_0^t(\|\nabla^2 u\|_2^2+\|\nabla^2 H\|_2^2)ds,$$
$$\Phi(t)=C_0^2\sup_{0\le s\le t}(\|\nabla u\|_2^2+\|\nabla H\|_2^2).$$

In view of the regularities of u and H, one can easily check that both $E(t)$ and $\Phi(t)$ are continuous functions on $[0,T]$. By Lemma 2.3, there is a positive constant C_*, such that

$$E(t)\le 2(\|\nabla u_0\|_2^2+\|\nabla H_0\|_2^2)+C_*[\Phi(t)^{1/2}+\Phi(t)]E(t). \tag{2.9}$$

We take

$$\varepsilon_0=\min\left\{\frac{1}{32C_*},\frac{1}{128C_*^2}\right\},$$

and suppose that

$$C_0^2(\|\nabla u_0\|_2^2+\|\nabla H_0\|_2^2)\le\varepsilon_0.$$

We claim that

$$\Phi(t)<\min\left\{\frac{1}{4C_*},\frac{1}{16C_*^2}\right\},\qquad 0\le t\le T.$$

Otherwise, by the continuity and monotonicity of $\Phi(t)$, there is $T_0\in(0,T]$, such that

$$\Phi(T_0)=\min\left\{\frac{1}{4C_*},\frac{1}{16C_*^2}\right\}. \tag{2.10}$$

On account of (2.10), it follows from (2.9) that

$$E(T_0)\le 2(\|\nabla u_0\|_2^2+\|\nabla H_0\|_2^2)+\frac12E(T_0),$$

and thus

$$E(T_0)\le 4(\|\nabla u_0\|_2^2+\|\nabla H_0\|_2^2).$$

Recalling the definition of $E(t)$ and $\Phi(t)$, we deduce from the above inequality that

$$\Phi(T_0)\le C_0^2E(T_0)\le 4C_0^2(\|\nabla u_0\|_2^2+\|\nabla H_0\|_2^2)\le 4\varepsilon_0=\min\left\{\frac{1}{8C_*},\frac{1}{32C_*^2}\right\},$$

contradicting to (2.10). This contradiction implies that the claim is true.

By the aid of the claim we proved in the above, it follows from (2.9) that

$$E(t) \leq 4(\|\nabla u_0\|_2^2 + \|\nabla H_0\|_2^2), \qquad 0 \leq t \leq T,$$

completing the proof. $\square$

Now, we can give the proof of Theorem 1.1.

Proof of Theorem 1.1: Let ε_0 be the constant stated in Lemma 2.4 and suppose that the initial data satisfies

$$(\|\sqrt{\rho_0}u_0\|_2^2 + \|H_0\|_2^2)(\|\nabla u_0\|_2^2 + \|\nabla H_0\|_2^2) \leq \varepsilon_0.$$

By Lemma 2.1, there is a unique strong solution (ρ, u, H) to system (1.1)–(1.6). Extend such local solution to the maximal existence time interval $[0, T_*)$. We will prove that $T_* = \infty$. Suppose, by contradiction, that $T_* < \infty$. By Lemma 2.1, the time T_* can be characterized as follows

$$\int_0^T \|(\nabla u, \nabla H)\|_2^8 dt < \infty, \quad \text{for all} \quad 0 < T < T_*,$$

and

$$\int_0^{T_*} \|(\nabla u, \nabla H)\|_2^8 dt = \infty. \tag{2.11}$$

By Lemma 2.4, for any $0 < T < T_*$, there holds

$$\sup_{0 \leq t \leq T} (\|\nabla u\|_2^2 + \|\nabla H\|_2^2) \leq 4(\|\nabla u_0\|_2^2 + \|\nabla H_0\|_2^2),$$

and thus

$$\sup_{0 \leq t < T_*} (\|\nabla u\|_2^2 + \|\nabla H\|_2^2) \leq 4(\|\nabla u_0\|_2^2 + \|\nabla H_0\|_2^2),$$

which implies

$$\int_0^{T_*} \|(\nabla u, \nabla H)\|_2^8 dt \leq 4(\|\nabla u_0\|_2^2 + \|\nabla H_0\|_2^2)T_* < \infty,$$

contradicting to (2.11). This contradiction provides us that $T_* = \infty$, and thus we obtain a global strong solution. The proof is complete.

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